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A model of an ionizing current sheet

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Abstract. The steady-state structure of an ionizing current layer ('sheet') propagating into a cold unionized gas is examined using a collision-free self-consistent field approach. The results parallel those for a collision-dominated plasma; in particular, it is found that the magnetic field may not reverse through the layer. The structure of the sheet is not steady-state unless there is a large ambient magnetic field, or collisions destroy the coherence of the particles' motion after they leave the 'sheet'. A sheet propagating into a strong magnetic field with long ion mean free path is shown to behave like a snow-plough.

1. Introduction

Many important pulsed plasma devices accelerate the gas by the interaction between currents in the plasma and magnetic fields, which may be generated externally or by the current in the discharge. In these experiments it is found that the current flows in a finite region which separates the ambient cold gas and magnetic field from a region of higher field strength and hot flowing gas (figure 1); the region in which the

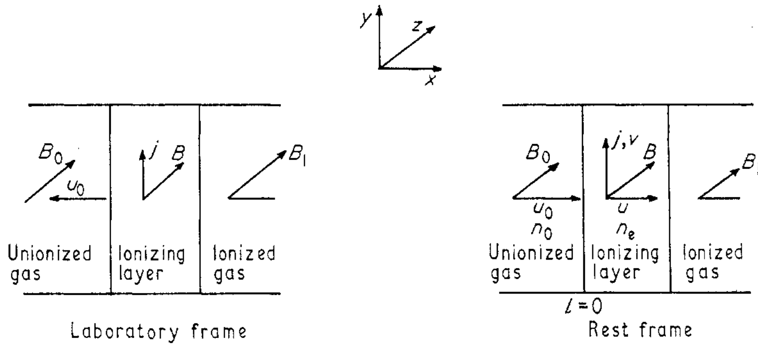

Figure 1. The structure of the ionizing current sheet is shown diagramatically in both the laboratory frame, in which the unionized gas is initially at rest, and the rest frame of the sheet.

gas is heated and accelerated is usually called the current sheet. In this note we shall consider only planar current sheets with the added assumption that the motion is time invariant.

As the current is primarily carried by the electrons in the plasma, there must be an interaction force between the electrons and the ions which transfers the momentum necessary to accelerate the heavy particles. At high densities this interaction is provided by collisions, and the structure of the sheet may be described in terms of an ionizing shock (Kunkel and Gross 1962, Chu 1964, Taussig 1965). However, at low densities this simple model is no longer adequate as the electron motion becomes collision-free; the electrons are now tightly bound to the magnetic field, which is stationary in the rest frame of the sheet. As a result of their initial velocity the ions overshoot the electrons and produce a space-charge separation electric field which accelerates the ions and, by virtue of the $\boldsymbol{E}\times\boldsymbol{B}$ electron drift, also drives the current in the sheet. Rosenbluth (1957) has investigated this problem in the case in which both electron and ion motions are collision-free and the ambient gas is initially ionized. It is found that, in the rest frame of the sheet, as a result of the field structure the ions and electrons move through the sheet with a progressivly decreasing speed until, at the back of the sheet, the ions are brought to rest, before retracing their path in reverse and leaving the sheet with the same speed as that with which they entered (in the rest frame) but in the opposite direction; the particles are thus reflected by the sheet. Kulinski (1968) has also considered the structure of the collision-free sheet. By including the effects of ionization and using an arbitrary (and rather unrealistic) current distribution he showed that particles may be reflected, transmitted or trapped by the sheet under various conditions. In this note we extend Rosenbluth's analysis to consider the steady-state structure of an ionizing sheet with a self-consistent field distribution. As in the case of the collision-dominated sheet it is found that a necessary condition for the sheet to form is the presence of an ambient magnetic field (Burton 1968).

2. Self-consistent field calculation

In the rest frame of the front consider cold unionized gas entering the front at $x = 0$ with velocity u_0 in the $+x$ direction and number density n_0. We assume that the particles are ionized once only at a constant rate

$$\frac{\mathrm{d}n_\mathrm{n}}{\mathrm{d}x} = -\frac{n_0}{\lambda} \tag{1}$$

in a distance λ which is less than the thickness of the sheet, where n_n is the density of unionized neutrals.

The subsequent motion of the ions and electrons is assumed throughout to be collision-free. The equations of motion for the velocity components in the x and y directions (u and v respectively) can be integrated to give

$$u = \left\{\frac{2e}{m}\int_{x'}^{x} E\,\mathrm{d}x - \left(\frac{e}{m}\right)^2\left(\int_{x'}^{x}\frac{B\,\mathrm{d}x}{c}\right)^2 + u_0^2\right\}^{1/2} \tag{2}$$

$$v = -\frac{e}{m}\int_{x'}^{x}\frac{B\,\mathrm{d}x}{c} \tag{3}$$

where x' is the point of ionization. Since the sheet is uniform in the yz plane, the only allowed field components are the electric field E in the x direction and the magnetic field B in the z direction. If the ion current is neglected, the magnetic field is given by

$$\frac{\mathrm{d}B}{\mathrm{d}x} = -\frac{4\pi j_\mathrm{e}}{c} = \frac{4\pi e n_\mathrm{e} v_\mathrm{e}}{c}. \tag{4}$$

The quasi-neutral condition $n_\mathrm{i} \simeq n_\mathrm{e}$ (Rosenbluth 1957) is satisfied by

$$u_\mathrm{i} \simeq u_\mathrm{e}$$

yielding the self-consistent fields condition

$$\int_{x'}^{x} E\,\mathrm{d}x \simeq -\frac{e}{2m_e}\left(\int_{x'}^{x}\frac{B\,\mathrm{d}x}{c}\right)^2 \tag{5}$$

provided B is a sufficiently rapidly varying function of x.

If the electron mean free path is greater than the thickness of the sheet, the particles enter the sheet and after ionization are reflected at the back in the $-x$ direction, as discussed earlier. In this case the magnetic field is given by equations (3) and (4).

In the region $0 < x < \lambda$ the field is given by

$$\frac{\mathrm{d}B}{\mathrm{d}x} = \frac{4\pi e n_0 u_0}{\lambda c}\int_0^x \frac{v_e(x')}{u_i(x')}\,\mathrm{d}x' + \frac{4\pi e n_0 u_0}{\lambda c}\int_0^\lambda \frac{v_e(x')}{u_i(x')}\,\mathrm{d}x'. \tag{6}$$

A useful approximate form obtained by setting $u_i(x') = u_0$ in equation (6) is valid in the range $B_0 < B \lesssim 0{\cdot}5\,B_1$, where B_1 is the magnetic field at the back of the sheet. As the magnetic field increases rapidly at the back of the sheet, this asymptotic form is valid over most of the sheet. Therefore

$$\frac{\mathrm{d}B}{\mathrm{d}x} = \frac{4\pi e n_0}{\lambda c}\int_0^x v_e(x')\,\mathrm{d}x' + \frac{4\pi e n_0}{\lambda c}\int_0^\lambda v_e(x')\,\mathrm{d}x'. \tag{7}$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x v_e(x')\,\mathrm{d}x' = \frac{e}{m_e}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\left(\int_{x'}^{x}\frac{B\,\mathrm{d}x}{c}\right)\mathrm{d}x' = \frac{e}{m_e c}\,xB. \tag{8}$$

Thus (8) gives

$$\frac{\mathrm{d}^2B}{\mathrm{d}x^2} = \frac{4\pi e^2 n_0}{m_e \lambda c^2}(x+\lambda)B = \beta^3(x+\lambda)B. \tag{9}$$

In the region $\lambda < x$ it is easily shown that, in the asymptotic limit,

$$\frac{\mathrm{d}^2B}{\mathrm{d}x^2} = \frac{8\pi e^2 n_0}{m_e c^2}B = 2\beta^3\lambda B. \tag{10}$$

The particles are reflected at a point X such that

$$u_i^2 \simeq u_0^2 - \frac{e^2}{m_e m_i}\left(\int_{x'}^{X}\frac{B\,\mathrm{d}x}{c}\right)^2 = 0. \tag{11}$$

It is easily shown that if $\lambda < X$ all the particles are reflected at nearly the same point, owing to the rapid variation of B with x. The gradients at $x = 0$ and $x = \lambda$ are given by

$$\left(\frac{\mathrm{d}B}{\mathrm{d}x}\right)_0 = -\beta^3\int_0^\lambda(\lambda - x)B\,\mathrm{d}x,\quad \left(\frac{\mathrm{d}B}{\mathrm{d}x}\right)_\lambda = 2\beta^3\int_0^\lambda xB\,\mathrm{d}x. \tag{12}$$

Equation (9) is Stokes' equation which has analytic solutions in terms of the Airy integral functions (Antosiewicz 1964):

$$B = a\mathrm{Ai}\{\beta(x+\lambda)\} + b\mathrm{Bi}\{\beta(x+\lambda)\} \tag{13}$$

where a and b are chosen such that the initial field at $x = 0$ is B_0 and equation (12) is satisfied.

The solution to equation (9) can take two forms (figure 2); the one taken depends on the values of the parameters λ and β. From (12) and (13) the constants a and b are easily shown to be given by

$$\frac{a}{b} = -\frac{\mathrm{Bi}'(\beta\lambda)-\frac{1}{2}\mathrm{Bi}'(2\beta\lambda)+\beta\lambda\int_{\beta\lambda}^{2\beta\lambda}\mathrm{Bi}(z)\,\mathrm{d}z}{\mathrm{Ai}'(\beta\lambda)-\frac{1}{2}\mathrm{Ai}'(2\beta\lambda)+\beta\lambda\int_{\beta\lambda}^{2\beta\lambda}\mathrm{Ai}(z)\,\mathrm{d}z}. \tag{14}$$

Since $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are monotonic for $z > 0$, and $\mathrm{Ai}(z) \to 0$ and $\mathrm{Bi}(z) \to +\infty$ as $z \to \infty$, we see that, if $b > 0$, the magnetic field is always in the same direction whereas, if $b < 0$, the field reverses (figure 2).

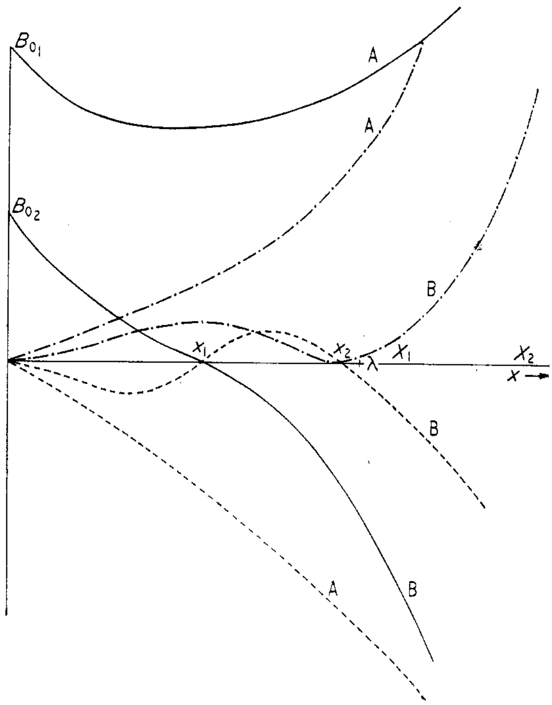

Figure 2. Diagrammatic plot of the parameters in current sheets in which the magnetic field is unidirectional (A) and reverses (B). It is shown that solutions with reversed field cannot satisfy the boundary conditions (equation (12)) and do not represent a steady-state sheet. Full curve, B; chain curve, $(\int B\,\mathrm{d}x)^2$; broken curve, E. $(\int B\,\mathrm{d}x)^2_{\mathrm{ref}} = \dfrac{m_e m_i c^2 u_0{}^2}{e^2}$.

From the asymptotic representations of the Airy functions and their monotonic nature for positive real moduli (Antosiewicz 1964) it can be shown that

$$\frac{a}{b} > 0 \qquad \text{and} \qquad \lim_{\beta\lambda\to\infty}\frac{a}{b} = 0$$

so that $b > 0$ and solutions with field reversal are not allowed, a result equivalent to that found by Chu (1964) for collision-dominated sheets.

At $x = 0$ there is an electron current in the $-y$ direction owing to the reflected particles ionized at $x > 0$. In the absence of collision, the magnetic field will increase along the $-x$ direction owing to this current, and the steady-state nature of the sheet may be lost. It may, however, be retained in two somewhat artificial models, which have physical importance. In the first, collisions rapidly destroy the current at $x < 0$. As the magnetic field in this region is small, the electron Hall parameter is small, and this condition may be satisfied in many typical experimental conditions; in these cases the reverse current may be neglected. The structure of the sheet is then described by the equations (2) and (9) as above. Also, if $\beta\lambda$ is small, the reverse current is small and the non-steady nature of the front may be neglected. Matching the solution in this case for $x < \lambda$: equation (9) to that for $x > \lambda$: equation (10), simply gives the asymptotic solution for the standard steady Rosenbluth sheath with

$$B = B_0 \cosh \alpha x \tag{15}$$

where $\alpha^2 = 8\pi n_0 e^2/m_e c^2$. For values of $\beta\lambda > 0$ the solution to equation (10) must be matched to that of (9) such that B and $\mathrm{d}B/\mathrm{d}x$ are continuous.

3. Extreme collisionless case

If the ion collision mean free path $\lambda_i \gg X$ the reflection length, the magnetic field at $x < 0$ will be due to the current carried by the electrons as they leave the sheet. In the extreme case a second sheath will also be established in the reverse direction for $x < 0$. This may be simply investigated by an extension of the Rosenbluth model by considering a simple model of the ionization region. We assume that at $x = 0$ a flux of particles $n_0 u_0$ is ionized.

For $x > 0$ the current density at x is given by

$$j = n_0 e v_e(x) \frac{\cosh\{(X-x)/\lambda_i\}}{\sinh(2X/\lambda_i)} \exp(X/\lambda_i) \tag{16}$$

since, provided $\lambda_e > X$, it is ion collisions that will predominantly determine the structure of the sheet; the electrons act almost as mass-free particles, whereas the ion motion is inertia-limited. Thus an electron collision in an established sheet does not destroy the structure of the fields. However, an ion collision destroys the coherence of the motion and removes the ion from the sheet.

With equation (16) for the current density, equation (4) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^x B\,\mathrm{d}x = \frac{4\pi n_0 e^2}{m_e c^2}\left(\int_0^x B\,\mathrm{d}x\right)\exp(X/\lambda_i)\frac{\cosh\{(X-x)/\lambda_i\}}{\sinh(2X/\lambda_i)}. \tag{17}$$

Expanding the cosh term it is easily shown that if $\lambda_i \gg X$ this reduces to a simple Rosenbluth sheet described by equation (15) with a flux of particles at $x = 0$ of density

$$n_r = \frac{1}{2}\frac{n_0 \exp(X/\lambda_i)}{\sinh(2X/\lambda_i)}. \tag{18}$$

In a similar manner the structure of the sheet at $x < 0$ is also described by equation (15) but in this case the particle density is

$$n_l = \frac{1}{2}\frac{n_0 \exp(-X/\lambda_i)}{\sinh(2X/\lambda_i)}. \tag{19}$$

The magnetic field at which reflection occurs can be shown to be given by

$$B^2 - B_0^2 = \frac{2\alpha^2 m_i m_e u_0^2 c^2}{e^2}. \tag{20}$$

The total magnetic field change across a sheet of this type is thus

$$B_r^2 - B_i^2 = \frac{2m_i m_e u_0^2}{e^2}\left\{\frac{1}{2}\frac{\exp(X/\lambda_i)}{\sinh(2X/\lambda_i)} - \frac{1}{2}\frac{\exp(-X/\lambda_i)}{\sinh(2X/\lambda_i)}\right\}\frac{8\pi n_0 e^2}{m_e} = \frac{8\pi n_0 u_0^2 m_i}{\cosh(X/\lambda_i)} \tag{21}$$

as expected. It should be noticed, however, that this steady-state sheet can only be established if the front propagates into an ambient magnetic field

$$B_l > \left\{\frac{8\pi m_i n_0 u_0^2 \exp(-X/\lambda_i)}{\sinh(2X/\lambda_i)}\right\}^{1/2}$$

which may be comparable with the field behind the sheet. The origin of the magnetic field B_l into which the sheet propagates is arbitrary. In a typical experiment it would be provided by an external set of field coils. However, in any idealized system with a long ion mean free path, it may be imagined that the 'left-hand' field B_l may result from the propagation of an electromagnetic wave carrying B_l as the reverse current is established in the region $x < 0$.

The particles in this sheet will be trapped until their motion is destroyed by a collision. They may leave the sheet in either the $+x$ or $-x$ direction, depending simply on the statistics of the problem, with a velocity less than u_0. However, from (21) we can see that if $X \ll \lambda_i$ the average speed of the particles after collision is zero in the rest frame, since the overall momentum change is just sufficient to accelerate the particles to the velocity of the sheet, and the sheet acts like a snowplough.

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